

# A note on subgaussian estimates for linear functionals on convex bodies

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## Abstract

We give an alternative proof of a recent result of Klartag on the existence of almost subgaussian linear functionals on convex bodies. If  $K$  is a convex body in  $\mathbb{R}^n$  with volume one and center of mass at the origin, there exists  $x \neq 0$  such that

$$|\{y \in K : |\langle y, x \rangle| \geq t \|\langle \cdot, x \rangle\|_1\}| \leq \exp(-ct^2 / \log^2(t+1))$$

for all  $t \geq 1$ , where  $c > 0$  is an absolute constant. The proof is based on the study of the  $L_q$ -centroid bodies of  $K$ . Analogous results hold true for general log-concave measures.

## 1 Introduction

The purpose of this note is to provide an alternative proof of a recent result of Klartag (see [9]) on the existence of almost subgaussian linear functionals on convex bodies. Let  $K$  be a convex body in  $\mathbb{R}^n$  with volume  $|K| = 1$  and center of mass at the origin. Let  $\psi : [0, \infty) \rightarrow [0, \infty)$  be a convex, increasing function with  $\psi(0) = 0$ . For every bounded measurable function  $f : K \rightarrow \mathbb{R}$ , define

$$(1.1) \quad \|f\|_\psi = \inf \left\{ t > 0 : \int_K \psi(|f(x)|/t) dx \leq 1 \right\}.$$

We will be interested in the  $\psi_\alpha$ -norm of linear functionals  $y \mapsto \langle y, x \rangle$  on  $K$ , where  $1 \leq \alpha \leq 2$  and  $\psi_\alpha(t) = e^{t^\alpha} - 1$ . We say that  $x \neq 0$  defines a  $\psi_\alpha$ -direction for  $K$  with constant  $B > 0$  if

$$(1.2) \quad \|\langle \cdot, x \rangle\|_{\psi_\alpha} \leq B \|\langle \cdot, x \rangle\|_1.$$

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It is not hard to check that this holds true if and only if

$$(1.3) \quad \|\langle \cdot, x \rangle\|_q \leq cBq^{1/\alpha} \|\langle \cdot, x \rangle\|_1$$

for every  $q \geq 1$ , where  $c > 0$  is an absolute constant. By Borell's lemma (see [13], Appendix III), there exists an absolute constant  $C > 0$  such that if  $K$  is a convex body in  $\mathbb{R}^n$ , then every  $x \neq 0$  is a  $\psi_1$ -direction for  $K$  with constant  $C$ .

The study of  $\psi_2$ -directions for linear functionals on convex bodies is motivated by the study of isotropic convex bodies and Bourgain's approach to the isotropic constant problem. A convex body  $K$  in  $\mathbb{R}^n$  is called isotropic if it has volume  $|K| = 1$ , center of mass at the origin, and there exists a constant  $L_K > 0$  such that

$$(1.4) \quad \int_K \langle y, \theta \rangle^2 dy = L_K^2$$

for every  $\theta \in S^{n-1}$ . Every convex body with center of mass at the origin has a linear image which is isotropic (see [12]). This image is unique up to orthogonal transformations, and hence, the isotropic constant  $L_K$  is well-defined for the linear class of  $K$ . The isotropic constant problem asks if there exists an absolute constant  $C > 0$  such that  $L_K \leq C$  for every isotropic convex body in any dimension. One can easily see that  $L_K = O(\sqrt{n})$  for every  $K$ . Uniform boundedness of  $L_K$  is known for some classes of bodies: unit balls of spaces with 1-unconditional basis, zonoids and their polars, etc. Bourgain (see [4]) proved that  $L_K = O(\sqrt[4]{n} \log n)$  and, very recently, Klartag (see [8]) improved this bound to  $L_K = O(\sqrt[4]{n})$ . Moreover, in [5] Bourgain proved that if every  $x \neq 0$  is a  $\psi_2$ -direction for  $K$  with constant  $B$ , then  $L_K$  is bounded by  $cB \log(B + 1)$ .

A question of Milman, related to this line of thought, is whether, for every isotropic convex body  $K$  in  $\mathbb{R}^n$ , most  $\theta \in S^{n-1}$  define a  $\psi_2$ -direction for  $K$  with a "good" constant (for example, logarithmic in  $n$ ). Until recently, it was not known if there exists an absolute constant  $C > 0$  such that every isotropic convex body has at least one  $\psi_2$ -direction with constant  $C$ . Some positive results are known for special classes of convex bodies. Bobkov and Nazarov (see [2] and [3]) have proved that if  $K$  is an isotropic 1-unconditional convex body, then  $\|\langle \cdot, x \rangle\|_{\psi_2} \leq c\sqrt{n}\|x\|_\infty$  for every  $x \neq 0$ . This shows that the diagonal direction is a  $\psi_2$ -direction. For the class of zonoids, the existence of good  $\psi_2$ -directions was established in [14]. Another partial result, which gives more information in the case of isotropic convex bodies with "small diameter", was obtained in [15]: If  $K \subseteq (\gamma\sqrt{n}L_K)B_2^n$  for some  $\gamma > 0$ , then

$$(1.5) \quad \sigma(\theta \in S^{n-1} : \|\langle \cdot, \theta \rangle\|_{\psi_2} \geq c_1\gamma t L_K) \leq \exp(-c_2\sqrt{n}t^2/\gamma)$$

for every  $t \geq 1$ , where  $\sigma$  is the rotationally invariant probability measure on  $S^{n-1}$  and  $c_1, c_2 > 0$  are absolute constants.

Klartag (see [9]) gave a positive answer to this question, showing that every isotropic convex body admits at least one almost subgaussian linear functional. Our aim is to give a second (short) proof of this fact.

**Theorem 1.1.** *Let  $K$  be an isotropic convex body in  $\mathbb{R}^n$ . There exists  $x \neq 0$  such that*

$$(1.6) \quad |\{y \in K : |\langle y, x \rangle| \geq t \|\langle \cdot, x \rangle\|_1\}| \leq \exp(-ct^2 / \log^\tau(t+1))$$

for all  $t \geq 1$ , where  $c, \tau > 0$  are absolute constants.

It is clear that if  $x$  defines a  $\psi_\alpha$ -direction for  $K$  and if  $T \in SL(n)$ , then  $T^*x$  defines a  $\psi_\alpha$ -direction (with the same constant) for  $T(K)$ . It follows that Theorem 1.1 provides almost subgaussian directions for every convex body: If  $K$  is a convex body in  $\mathbb{R}^n$  with volume one and center of mass at the origin, there exists  $x \neq 0$  such that (1.6) holds true for all  $t \geq 1$ .

The argument of Klartag is based on the study of the level sets of the logarithmic Laplace transform of log-concave functions. The argument we present here is based on the study of the  $L_q$ -centroid bodies of an isotropic convex body. This family of bodies was studied and used by the third named author in [15], and in particular in [16], where the following sharp dimension-dependent concentration of volume estimate was proved: There exists an absolute constant  $c > 0$  such that if  $K$  is an isotropic convex body in  $\mathbb{R}^n$ , then

$$(1.7) \quad |\{x \in K : \|x\|_2 \geq c\sqrt{n}L_K t\}| \leq \exp(-\sqrt{nt})$$

for every  $t \geq 1$ , where  $\|\cdot\|_2$  is the Euclidean norm. The tools which are developed in [16] allow us to give a very simple proof of Theorem 1.1. We present an argument which gives  $\tau = 2$ , i.e. the upper bound in (1.6) is  $\exp(-ct^2 / \log^2(t+1))$ .

**Notation.** We work in  $\mathbb{R}^n$ , which is equipped with a Euclidean structure  $\langle \cdot, \cdot \rangle$ . We denote by  $\|\cdot\|_2$  the corresponding Euclidean norm, and write  $B_2^n$  for the Euclidean unit ball, and  $S^{n-1}$  for the unit sphere. Volume is denoted by  $|\cdot|$ . If  $K$  is a convex body in  $\mathbb{R}^n$ , we set  $\overline{K} = K/|K|^{1/n}$ ; this is the dilation of  $K$  which has volume one. We write  $\sigma$  for the rotationally invariant probability measure on  $S^{n-1}$ . The Grassmann manifold  $G_{n,k}$  of  $k$ -dimensional subspaces of  $\mathbb{R}^n$  is equipped with the Haar probability measure  $\mu_{n,k}$ .

A convex body is a compact convex subset  $C$  of  $\mathbb{R}^n$  with non-empty interior. We say that  $C$  has center of mass at the origin if  $\int_C \langle x, \theta \rangle dx = 0$  for every  $\theta \in S^{n-1}$ . The support function  $h_C : \mathbb{R}^n \rightarrow \mathbb{R}$  of  $C$  is defined by  $h_C(x) = \max\{\langle x, y \rangle : y \in C\}$ . The mean width of  $C$  is defined by

$$(1.8) \quad w(C) = \int_{S^{n-1}} h_C(\theta) \sigma(d\theta).$$

The letters  $c, c', c_1, c_2$  etc. denote absolute positive constants which may change from line to line. We refer to the books [18], [13] and [17] for basic facts from the Brunn-Minkowski theory and the asymptotic theory of finite dimensional normed spaces.

## 2 Normalized $L_q$ -centroid bodies

Let  $K$  be a convex body of volume 1 in  $\mathbb{R}^n$ . For every  $q \geq 1$  we define the  $L_q$ -centroid body  $Z_q(K)$  of  $K$  by its support function:

$$(2.1) \quad h_{Z_q(K)}(x) = \|\langle \cdot, x \rangle\|_q := \left( \int_K |\langle y, x \rangle|^q dy \right)^{1/q}.$$

Since  $|K| = 1$ , we readily see that  $Z_1(K) \subseteq Z_p(K) \subseteq Z_q(K) \subseteq Z_\infty(K)$  for every  $1 \leq p \leq q \leq \infty$ , where  $Z_\infty(K) = \text{conv}\{K, -K\}$ . On the other hand, one has the reverse inclusions

$$(2.2) \quad Z_q(K) \subseteq \frac{cq}{p} Z_p(K)$$

for every  $1 \leq p < q < \infty$ , as a consequence of the  $\psi_1$ -behavior of  $y \mapsto \langle y, x \rangle$ . Observe that  $Z_q(K)$  is always symmetric, and  $Z_q(TK) = T(Z_q(K))$  for every  $T \in SL(n)$  and  $q \in [1, \infty]$ . Also, if  $K$  has its center of mass at the origin, then  $Z_q(K) \supseteq cZ_\infty(K)$  for all  $q \geq n$ , where  $c > 0$  is an absolute constant.

It should be mentioned that  $L_q$ -centroid bodies were introduced in [10] under a different normalization. Lutwak, Yang and Zhang (see [11] and [7] for a different proof) have established the  $L_q$  affine isoperimetric inequality

$$(2.3) \quad |Z_q(K)|^{1/n} \geq |Z_q(\overline{B}_2^n)|^{1/n} \geq c\sqrt{q/n}$$

for every  $1 \leq q \leq n$ , where  $c > 0$  is an absolute constant.

We will need upper estimates for the quermassintegrals of the  $L_q$ -centroid bodies of an isotropic convex body. These follow immediately from estimates on the projections of  $Z_q(K)$ , which are obtained in [16]. Fix  $1 \leq k \leq n$  and a  $k$ -dimensional subspace  $F$  of  $\mathbb{R}^n$ , and denote by  $E$  the orthogonal subspace of  $F$ . For every  $\phi \in S_F$ , define  $E(\phi) = \{y \in \text{span}\{E, \phi\} : \langle y, \phi \rangle \geq 0\}$ . By a theorem of K. Ball (see [1] and [12]), for every convex body  $K$  of volume one in  $\mathbb{R}^n$ , for every  $q \geq 0$  and every  $\phi \in F$ , the function

$$(2.4) \quad \phi \mapsto \|\phi\|_2^{1+\frac{q}{q+1}} \left( \int_{K \cap E(\phi)} |\langle y, \phi \rangle|^q dy \right)^{-\frac{1}{q+1}}$$

is a gauge function on  $F$  (see also [6] for the not necessarily symmetric case). If we denote by  $B_q(K, F)$  the convex body in  $F$  whose gauge function is defined by (2.4), then the volume of  $B_q(K, F)$  is given by

$$(2.5) \quad |B_q(K, F)| = |B_2^k| \int_{S_F} \left( \int_{K \cap E(\phi)} |\langle x, \phi \rangle|^q dx \right)^{\frac{k}{q+1}} d\sigma_F(\phi).$$

The following identity was proved in [16].

**Proposition 2.1.** *Let  $K$  be a convex body of volume 1 in  $\mathbb{R}^n$  and let  $1 \leq k \leq n-1$ . For every  $F \in G_{n,k}$  and every  $q \geq 1$  we have that*

$$(2.6) \quad P_F(Z_q(K)) = (k+q)^{1/q} |B_{k+q-1}(K, F)|^{1/k+1/q} Z_q(\overline{B}_{k+q-1}(K, F)).$$

Using this identity and exploiting (2.5) in order to estimate the volume of  $B_q(K, F)$ , one gets the following estimate (see [16]).

**Proposition 2.2.** *Let  $K$  be an isotropic convex body in  $\mathbb{R}^n$ . If  $F \in G_{n,k}$  and  $E = F^\perp$  then, for every  $q \in \mathbb{N}$  we have that*

$$(2.7) \quad P_F(Z_q(K)) \leq \frac{c(k+q)}{k} L_K Z_q(\overline{B}_{k+q-1}(K, F))$$

where  $c > 0$  is an absolute constant.

**Definition 2.3.** Let  $K$  be an isotropic convex body in  $\mathbb{R}^n$ . For every integer  $q \geq 1$  we define the *normalized  $L_q$ -centroid body  $K_q$*  of  $K$  by

$$(2.8) \quad K_q = \frac{1}{\sqrt{q} L_K} Z_q(K).$$

Since  $|Z_q(\overline{B}_{k+q-1}(K, F))| \leq |\overline{B}_{k+q-1}(K, F)| = 1$ , Proposition 2.2 shows that

$$(2.9) \quad |P_F(K_q)|^{1/k} \leq \frac{c(k+q)}{k\sqrt{q}} |Z_q(\overline{B}_{k+q-1}(K, F))|^{1/k} \leq \frac{c_1(k+q)}{k} \frac{\sqrt{k}}{\sqrt{q}} |B_2^k|^{1/k}$$

for every  $F \in G_{n,k}$ . If  $1 \leq k \leq q$ , this estimate takes the simpler form

$$(2.10) \quad |P_F(K_q)|^{1/k} \leq 2c_1 \frac{\sqrt{q}}{\sqrt{k}} |B_2^k|^{1/k}.$$

In particular, for every  $F \in G_{n,q}$  we have

$$(2.11) \quad |P_F(K_q)|^{1/k} \leq 2c_1 |B_2^k|^{1/k}.$$

A standard argument (based on the log-concavity of the quermassintegrals of  $P_F(K_q)$ ) implies that since (2.11) is true for every  $F \in G_{n,q}$ , it remains valid for every  $F \in G_{n,k}$ , where  $q \leq k \leq n$ . We summarize these observations in the next Theorem.

**Theorem 2.4.** *Let  $K$  be an isotropic convex body in  $\mathbb{R}^n$ . If  $1 \leq k, q \leq n$  are integers, and if  $F \in G_{n,k}$ , then*

$$(2.12) \quad |P_F(K_q)|^{1/k} \leq c_1 \max\{\sqrt{q/k}, 1\} |B_2^k|^{1/k},$$

where  $c_1 > 0$  is an absolute constant. In particular,

$$(2.13) \quad |K_q|^{1/n} \leq c_1 |B_2^n|^{1/n}.$$

The last ingredient of the proof is a consequence of the main result in [16]: from (1.7) it follows that

$$(2.14) \quad \left( \int_K \|y\|_2^q dy \right)^{1/q} \leq c\sqrt{n}L_K$$

for all  $1 \leq q \leq \sqrt{n}$ . Since

$$(2.15) \quad w(Z_q(K)) \leq \left( \int_{S^{n-1}} \int_K |\langle y, \theta \rangle|^q dy \sigma(d\theta) \right)^{1/q} \leq \left( \frac{C\sqrt{q}}{\sqrt{n}} \int_K \|y\|_2^q dy \right)^{1/q}$$

for all  $1 \leq q \leq n$ , we have the following Lemma.

**Lemma 2.5.** *Let  $K$  be an isotropic convex body in  $\mathbb{R}^n$ . If  $1 \leq q \leq \sqrt{n}$ , then*

$$(2.16) \quad w(K_q) \leq C,$$

where  $C > 0$  is an absolute constant.

**Remark 2.6.** Without using Lemma 2.5, which fully exploits the results of [16], we can prove Theorem 1.1 with  $\tau = 2 + \epsilon$  for any  $\epsilon > 0$ .

### 3 Covering numbers of $K_q$

Let  $N(K_q, sB_2^n)$  denote the minimal number of translates of  $sB_2^n$  whose union covers  $K_q$ . A standard way to estimate the covering number  $N(K_q, sB_2^n)$  is through the inequality

$$(3.1) \quad |tB_2^n| \cdot N(K_q, 2tB_2^n) \leq |K_q + tB_2^n|,$$

which is valid for every  $t > 0$ . We will use our information on the projections of  $K_q$  in order to give an upper bound for  $|K_q + tB_2^n|$ .

**Proposition 3.1.** *Let  $K$  be an isotropic convex body in  $\mathbb{R}^n$ . For every  $1 \leq q \leq n$  and every  $t > 0$ , we have that*

$$(3.2) \quad N(K_q, 2tB_2^n) \leq \exp \left( C \frac{\sqrt{qn}}{\sqrt{t}} + C \frac{n}{t} \right),$$

where  $C > 0$  is an absolute constant.

*Proof.* From the classical Steiner's formula we know that

$$(3.3) \quad |K_q + tB_2^n| = \sum_{k=0}^n \binom{n}{k} W_{[n-k]}(K_q) t^{n-k}$$

for all  $t > 0$ , where  $W_{[n-k]}(K_q)$  is the mixed volume  $V_k(K_q) = V(K_q; k, B_2^n; n-k)$  (see [18]).

We will use Kubota's integral formula to express  $W_{[n-k]}(K_q)$  as an average of the volumes of the  $k$ -dimensional projections of  $K_q$ : for every  $1 \leq k \leq n-1$  we have

$$(3.4) \quad W_{[n-k]}(K_q) = \frac{|B_2^n|}{|B_2^k|} \int_{G_{n,k}} |P_F(K_q)| d\mu_{n,k}(F).$$

Using (3.3), (3.4) and the estimates from Theorem 2.4, we can write

$$(3.5) \quad |K_q + tB_2^n| \leq |B_2^n| \sum_{k=0}^n \binom{n}{k} \left( c_1 \max\{\sqrt{q/k}, 1\} \right)^k t^{n-k}.$$

Then, (3.1) shows that

$$(3.6) \quad N(K_q, 2tB_2^n) \leq \sum_{k=0}^q \left( \frac{c_2 n \sqrt{q}}{k^{3/2} t} \right)^k + \sum_{k=q+1}^n \left( \frac{c_2 n}{kt} \right)^k.$$

Observe that for  $1 \leq k \leq q$  we have

$$(3.7) \quad \left( \frac{c_2 n \sqrt{q}}{k^{3/2} t} \right)^k \leq \left( \frac{c_2 n q}{k^2 t} \right)^k \leq \frac{(c_3 \sqrt{nq/t})^{2k}}{(2k)!},$$

while, for  $q \leq k \leq n$  we have

$$(3.8) \quad \left( \frac{c_2 n}{kt} \right)^k \leq \frac{(c_4 n/t)^k}{k!}.$$

It follows that

$$(3.9) \quad N(K_q, 2tB_2^n) \leq \exp \left( c_3 \frac{\sqrt{qn}}{\sqrt{t}} \right) + \exp \left( c_4 \frac{n}{t} \right),$$

and the result follows.  $\square$

**Remark 3.2.** The proof actually gives  $N(K_q, 2tB_2^n) \leq \exp \left( C \frac{n^{2/3} q^{1/3}}{t^{2/3}} + C \frac{n}{t} \right)$  for every  $t > 0$ , but this would play no role in the proof of the main result.

## 4 Proof of the Theorem

Let  $K$  be an isotropic convex body in  $\mathbb{R}^n$ . Consider the convex body

$$(4.1) \quad T = \text{conv} \left( \bigcup_{i=1}^{\lfloor \log_2 n \rfloor} \frac{1}{i} K_{2^i} \right).$$

We will use the following standard fact.

**Lemma 4.1.** *Let  $A_1, \dots, A_s$  be subsets of  $RB_2^n$ . For every  $t > 0$  we have that*

$$(4.2) \quad N(\text{conv}(A_1 \cup \dots \cup A_s), 2tB_2^n) \leq \left(\frac{cR}{t}\right)^s \prod_{i=1}^s N(A_i, tB_2^n).$$

*Sketch of the proof.* For  $i = 1, \dots, s$ , let  $N_i$  be a subset of  $\mathbb{R}^n$  with cardinality  $|N_i| = N(A_i, tB_2^n)$ , so that  $A_i \subseteq \bigcup_{x_i \in N_i} (x_i + tB_2^n)$ . Let  $B_1^s$  denote the unit ball of  $\ell_1^s$  and fix  $Z \subseteq B_1^s$  of minimal cardinality, so that  $B_1^s \subseteq \bigcup_{z \in Z} (z + (t/R)B_1^n)$ . It is well-known that  $|Z| \leq (cR/t)^s$ , where  $c > 0$  is an absolute constant. Consider the set  $N = \{w = z_1x_1 + \dots + z_sx_s : x_i \in N_i, z = (z_1, \dots, z_s) \in Z\}$ . Then,  $\text{conv}(A_1 \cup \dots \cup A_s) \subseteq \bigcup_{w \in N} (w + 2tB_2^n)$ .  $\square$

Let  $s = \lfloor \log_2 n \rfloor$  and  $m = \lfloor \log_2(\sqrt{n}) \rfloor \simeq s/2$ . We apply Lemma 4.1 with  $A_i = \frac{1}{i}K_{2^i}$ ,  $1 \leq i \leq s$ , and  $t = 1$ . Observe that  $A_i \subseteq c_1\sqrt{n}B_2^n$  for all  $i \leq s$  (to see this, recall the known fact that if  $K$  is an isotropic convex body in  $\mathbb{R}^n$ , then  $K \subseteq (cnL_K)B_2^n$ ). Using Sudakov's inequality (see [17]) and Lemma 2.5 to estimate  $N(A_i, B_2^n)$  for  $i \leq m$ , and using the entropy estimates of Section 3 to estimate  $N(A_i, B_2^n)$  for  $m < i \leq s = \lfloor \log_2 n \rfloor$ , we may write

$$\begin{aligned} N(T, B_2^n) &\leq (c_2\sqrt{n})^{\lfloor \log_2 n \rfloor} \left[ \prod_{i=1}^{\lfloor \log_2 n \rfloor} N(K_{2^i}, iB_2^n) \right] \\ &\leq e^{c_3n} \exp \left( C\sqrt{n} \sum_{i=s+1}^{\lfloor \log_2 n \rfloor} 2^{i/2} \right) \times \exp \left( Cn \cdot \left( \sum_{i=1}^m \frac{1}{i^2} + \sum_{i=m+1}^{2m} \frac{1}{i} \right) \right) \\ &\leq e^{cn}. \end{aligned}$$

It follows that  $|T| \leq |CB_2^n|$ , where  $C > 0$  is an absolute constant. Therefore, there exists  $x \neq 0$  such that

$$(4.3) \quad h_T(x) \leq C\|x\|_2,$$

and hence,

$$(4.4) \quad \|\langle \cdot, x \rangle\|_{2^i} \leq C 2^{i/2} i L_K \|x\|_2$$

for every  $i = 1, 2, \dots, \lfloor \log_2 n \rfloor$ . This easily implies the following.

**Theorem 4.2.** *Let  $K$  be an isotropic convex body in  $\mathbb{R}^n$ . There exists  $\theta \in S^{n-1}$  such that*

$$(4.5) \quad \|\langle \cdot, \theta \rangle\|_q \leq C\sqrt{q} \log q \|\langle \cdot, \theta \rangle\|_2$$

for every  $q \geq 2$ , where  $C > 0$  is an absolute constant.

A standard argument shows that Theorem 4.2 implies Theorem 1.1 (it is actually equivalent to Theorem 1.1 with  $\tau = 2$ ).



**Remark 4.3.** The proof of Theorem 4.2 carries over to the case of an arbitrary log-concave measure: the approach of [16] and all the arguments we have used in this note depend only on the Brunn–Minkowski theory. It follows that if  $\mu$  is an isotropic log-concave measure in  $\mathbb{R}^n$ , then there exists  $\theta \in S^{n-1}$  such that

$$(4.6) \quad \|\langle \cdot, \theta \rangle\|_{L^q(\mu)} \leq C\sqrt{q} \log q \|\langle \cdot, \theta \rangle\|_{L^2(\mu)}$$

for all  $2 \leq q \leq n$ , where  $C > 0$  is an absolute constant.

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